Spin effects in the effective quantum field theory of general relativity

2007 J. Phys. A: Math. Theor. 40 6973

(http://iopscience.iop.org/1751-8121/40/25/S48)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 6973–6978 doi:10.1088/1751-8113/40/25/S48

Spin effects in the effective quantum field theory of general relativity

Andreas Ross and Barry R Holstein

Department of Physics, University of Massachusetts, Amherst, MA 01003, USA

E-mail: andreasr@physics.umass.edu and holstein@physics.umass.edu

Received 31 October 2006, in final form 23 January 2007
Published 6 June 2007
Online at stacks.iop.org/JPhysA/40/6973

Abstract
The calculation of gravitational scattering of particles with various spin configurations in the framework of effective quantum field theory of general relativity is presented. We find that the long-range quantum and classical corrections to the spin-independent piece and the spin–orbit coupling piece of the scattering amplitude and the scattering potential are universal for different spin configurations, which leads us to propose this universality to be true for arbitrary spins. Furthermore, we give the leading corrections to the spin–spin coupling, and a comparison with scattering in QED is made.

PACS numbers: 03.70.+k, 04.20.−q, 12.20.−m

1. Introduction

It has long been known that general relativity is a non-renormalizable quantum field theory. However if treated as an effective theory, renormalization can be performed order by order and reliable predictions can be made at low energies, i.e., at energies much smaller than the Planck scale [1, 2]. What the effective field theory treatment accomplishes is to separate known physics associated with laboratory energy scales from unknown physics at very high energies—predictions at low energies are possible without knowledge of unknown high energy physics, which in our case would represent the UV completion of quantum gravity.

The effective action is constructed using the known symmetries and low energy degrees of freedom

$$S = \int \mathrm{d}^4x\, \sqrt{-g}(\mathcal{L}_{\mathrm{grav}} + \mathcal{L}_{\mathrm{matter}}), \tag{1}$$

where the purely gravitational part is

$$\mathcal{L}_{\mathrm{grav}} = \Lambda + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \cdots \tag{2}$$

with $\kappa^2 = 32\pi G \propto 1/M_{Pl}^2$, and the matter part is illustrated by a scalar field

$$\mathcal{L}_{\mathrm{matter}} = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 + \kappa^2\left(d_1 R^{\mu\nu} + d_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi + \kappa^2 d_3 R m^2\phi^2 + \cdots. \tag{3}$$

Since R and $R_{\mu\nu}$ each contain two powers of derivatives, it is seen that the effective action represents an expansion in p^2/M_{Pl}^2, where p are graviton momenta, so that reliable predictions are possible for energies much smaller than the Planck scale. Using the background field method, expanding around flat space—$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$—and fixing the gauge using the harmonic gauge fixing condition—$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h^\lambda_\lambda$—we can quantize the theory and derive Feynman rules. As we shall see, the leading corrections at long distances coming from non-local/non-analytic effects yield unique predictions independent of the unknown couplings c_i and d_i of the local Lagrangian. For a more detailed introduction to the effective field theory description of gravity see [3].

Within this framework we investigate the scattering of particles with various spins in the non-relativistic limit, which allows us to extract the leading long distance corrections to the scattering amplitude and to generate corrections to the Newtonian potential. For spinless particles this calculation has been performed by Bjerrum-Bohr, Donoghue and Holstein [4], and we summarize their work in the following section. In section 3 we extend the calculation to particles with spin. We find that the spin-independent pieces of the amplitude/potential are universal, i.e., they do not depend on the spins of the scattered particles, and we present new components that describe spin–orbit coupling and spin–spin coupling. The form of the spin–orbit pieces is also shown to be universal, and we expect such universalities to hold for arbitrary spins. For spin-1/2–spin-1/2 scattering, we compare our results to those of a paper by Kirilin [5] where the quantum pieces were calculated. In section 4 we draw a comparison to scattering in QED where similar universalities are found to occur.

2. Gravitational scattering of spinless particles

The tree level (one graviton exchange) amplitude for gravitational scattering of two particle with masses m_1 and m_2 can easily be calculated and reads

$$M = -\frac{4\pi G m_1 m_2}{q^2} + 4\pi G \xrightarrow{\mathrm{NR}} \frac{4\pi G m_1 m_2}{\vec{q}^2} + 4\pi G \tag{4}$$

in the non-relativistic limit—$v \ll c$—with q^2 representing the invariant momentum transfer squared. Note that only the first piece of equation (4) is non-analytic in q^2 and the amplitude has a single power of Newton's constant G. If we Fourier transform to coordinate space the Newtonian potential $V(r) = -Gm_1m_2/r$ is obtained from the non-analytic piece whereas the constant yields a delta function, which illustrates the fact that only pieces of the amplitude that are non-analytic in q^2 contribute to the long-range interaction.

The one-loop (two graviton exchange) calculation of the non-analytic component of the scattering amplitude yields $\mathcal{O}(G^2)$ corrections and is much more complicated. The Feynman diagrams involved can be found in [4], and the result for the non-analytic parts of the non-relativistic amplitude to 1-loop is

$$M = -\frac{4\pi G m_1 m_2}{q^2} + 6G^2 m_1 m_2(m_1+m_2)\frac{\pi^2}{\sqrt{-q^2}} - \frac{41}{5}G^2 m_1 m_2 \log - q^2. \tag{5}$$

The piece of this amplitude involving $(-q^2)^{-1/2}$ is of classical origin, while the $\log -q^2$ part is a quantum correction associated with zitterbewegung, i.e., the quantum piece is $\mathcal{O}(\hbar)$ whereas the classical component is $\hbar$ independent. To illustrate this feature, we display factors of $\hbar$ explicitly in the Fourier transform of the scattering amplitude, which gives the leading corrections to the Newtonian potential

$$V(r) = -\frac{Gm_1m_2}{r}\left(1 + 3\frac{G(m_1+m_2)}{r} + \frac{41}{10}\frac{G\hbar}{r^2} + \cdots\right). \tag{6}$$

The small expansion parameters of the corrections are $GM/r \propto (M/M_{Pl})(l_{Pl}/r)$ for the classical piece and $G\hbar/r^2 \propto (l_{Pl}/r)^2$ for the quantum piece. Since for macroscopic stellar objects such as stars $M \gg M_{Pl}$, the classical correction can become measurable even though $l_{Pl}/r \ll 1$, but unfortunately, the quantum corrections are tiny and there appears to be no hope to measure them.

An interesting feature is that the coefficient of the classical piece depends on the choice of coordinates. Our classical piece is the result for harmonic coordinates and would be different in a different gauge. The quantum piece however is gauge independent. This gauge dependence does not alter physical results, e.g., the perihelion of Mercury, which we could in principle compute. However, for such an endeavor, corrections proportional to v/c, which we have neglected, need to be taken into account and the equations of motion for which the potential is suitable would have to be identified. Clearly, the potential itself is not a physical observable—the concept of a potential in quantum field theories is a subtle one [6].

3. Gravitational scattering of particles with spin

The simplest case of scattering involving spin is the scattering of a spin-0 particle (mass m_1, incoming momentum p_1, outgoing momentum $p_1' = p_1 - q$) and a spin-1/2 particle (having mass m_2, incoming momentum p_2, outgoing momentum p_2'). At tree level the non-relativistic amplitude is

$$M = -\frac{4\pi G m_1 m_2}{q^2}\bar{u}(p_2')u(p_2) - \mathrm{i}\frac{8\pi G}{m_2 q^2}\epsilon_{\alpha\beta\gamma\delta}q^\alpha p_1^\beta p_2^\gamma S_2^\delta, \tag{7}$$

with the spin four-vector $S_2^\mu = \frac{1}{2}\bar{u}(p_2')\gamma_5\gamma^\mu u(p_2) \xrightarrow{NR} (0, \vec{S}_2)$ which reduces to the spin three-vector $\vec{S}_2 = \chi_f^\dagger \frac{\vec{\sigma}}{2}\chi_i$ in the non-relativistic limit. Our spinors are normalized as $\bar{u}(p)u(p) = 1$. It should be noted that pieces involving the spin four-vector yield only spin-dependent terms, while the $\bar{u}u$ component contains both spin-dependent as well as spin-independent pieces since $\bar{u}(p_2')u(p_2) \xrightarrow{NR} \chi_f^\dagger\chi_i - \frac{\mathrm{i}}{2m_2^2}\epsilon^{jkl}q^j p_2^k S_2^l$.

Performing the non-relativistic reduction of equation (7) and Fourier transforming, we find the potential in the centre-of-mass frame ($\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}, \vec{r} \equiv \vec{r}_1 - \vec{r}_2$):

$$V(r) = -\frac{G m_1 m_2}{r}\chi_f^\dagger\chi_i + \frac{2G\left(1 + \frac{3m_1}{4m_2}\right)\vec{L}\cdot\vec{S}_2}{r^3}, \tag{8}$$

with $\vec{L} = \vec{r}\times\vec{p}$. The first term is, of course, the spin-independent Newtonian potential while the second one represents the leading-order spin–orbit coupling, whose structure leads to the geodetic precession currently being measured by gravity probe B.

For spin-0–spin-1/2, scattering the calculation of the one-loop amplitude involves the same diagrams as in the spinless case, and the resulting correction to the scattering amplitude reads

$$\begin{aligned} M = {}& \left[6G^2 m_1 m_2(m_1+m_2)\frac{\pi^2}{\sqrt{-q^2}} - \frac{41}{5}G^2 m_1 m_2 \log -q^2\right]\bar{u}(p_2')u(p_2) \\ & + \left[\mathrm{i}\left(4\frac{m_1 m_2}{(m_1+m_2)^2}\frac{m_1 m_2}{\vec{p}^2} + 10\right)G^2\left(1 + \frac{3m_1}{4m_2}\right)\frac{\pi^2}{\sqrt{-q^2}}\right]\epsilon_{\alpha\beta\gamma\delta}q^\alpha p_1^\beta p_2^\gamma S_2^\delta \\ & - \mathrm{i}\frac{64G^2}{5m_2}\log -q^2 \epsilon_{\alpha\beta\gamma\delta}q^\alpha p_1^\beta p_2^\gamma S_2^\delta. \end{aligned} \tag{9}$$

The first feature we note is that the spin-independent corrections are identical to those found in the spinless case. Moreover, the classical correction to the spin–orbit coupling seen in the second line displays an intriguing behaviour, in that a part involves the factor $m_1m_2/\vec{p}^2$, which diverges in the threshold limit—$s \to s_0 = (m_1+m_2)^2$ and $\vec{p}^2 \to 0$. This piece arises from the reduction of the vector and tensor box integrals wherein the Gram determinant vanishes at threshold in the limit $\vec{q}^2 \ll \vec{p}^2$. In order to see why this piece is troublesome we take the Fourier transform of its non-relativistic reduction and generate its contribution to the potential

$$\begin{aligned} V_{NLO}^{SO} &= \frac{4G^2m_1^2m_2^2\left(1+\frac{3m_1}{4m_2}\right)\vec{L}\cdot\vec{S}_2}{(m_1+m_2)\vec{p}^2r^4} = -\frac{2m_1m_2}{(m_1+m_2)\vec{p}^2}V_{LO}^{SI}V_{LO}^{SO} \\ &= -\frac{Gm_1m_2/r}{\vec{p}^2/(2\mu)}V_{LO}^{SO}, \end{aligned} \tag{10}$$

where V_{LO}^{SI} and V_{LO}^{SO} are the spin-independent and spin–orbit components of the leading-order potential in equation (8). Since kinetic and potential energies are generally expected to be of the same order for systems that obey the virial theorem, this piece appears to be of the same order as the leading spin–orbit component! However, we have verified that equation (10) arises from the second Born approximation of the leading-order potential. If the goal is to define a potential to be used in a Schrödinger equation, then equation (10) must be subtracted from the potential in order to avoid double counting. Nevertheless, such terms *are* present in the scattering amplitude and they are $\hbar$ independent, which raises the question whether such terms could affect the modelling of gravitational wave signals from coalescing rotating black holes where spin-dependent post-Newtonian pieces play a role. One might think of such a term as a peculiarity of gravity, but the same behaviour is found for the spin-dependent classical pieces in electromagnetic scattering.

The quantum pieces of spin–orbit corrections to the spin-0–spin-1/2 scattering amplitude are well behaved and their contribution to the potential yields

$$V_{NLO,\mathrm{quantum}}^{SO}(r) = \frac{96G^2\vec{L}\cdot\vec{S}_2}{5\pi r^5} + \frac{m_1}{m_2}\frac{261G^2\vec{L}\cdot\vec{S}_2}{20\pi r^5}. \tag{11}$$

We have also calculated spin-0–spin-1 scattering, and the results have the same form and coefficients as the spin-0–spin-1/2 case, confirming the universality of corrections to the spin-independent and spin–orbit pieces. Furthermore, new *quadrupole* interaction terms appear and part of their classical piece also exhibits the interesting threshold behaviour proportional to $1/\vec{p}^2$. Detailed results will be given in [7].

The next case to consider is spin-1/2–spin-1/2 scattering where we find new spin–spin coupling terms, and where we again wish to confirm universality. At tree level the amplitude is

$$\begin{aligned} M = &-\frac{4\pi Gm_1m_2}{q^2}\bar{u}(p_1')u(p_1)\bar{u}(p_2')u(p_2) - \mathrm{i}\frac{8\pi G}{m_2q^2}\epsilon_{\alpha\beta\gamma\delta}q^\alpha p_1^\beta p_2^\gamma S_2^\delta\bar{u}(p_1')u(p_1) \\ &- \mathrm{i}\frac{8\pi G}{m_1q^2}\epsilon_{\alpha\beta\gamma\delta}q^\alpha p_1^\beta p_2^\gamma S_1^\delta\bar{u}(p_2')u(p_2) - \frac{4\pi G}{q^2}q\cdot S_1q\cdot S_,2 \end{aligned} \tag{12}$$

and the leading-order potential becomes (obvious factors of $\chi_f^\dagger \chi_i$ have been suppressed)

$$V(r) = -\frac{Gm_1m_2}{r} + \frac{2G\left(1+\frac{3m_2}{4m_1}\right)\vec{L}\cdot\vec{S}_1}{r^3} + \frac{2G\left(1+\frac{3m_1}{4m_2}\right)\vec{L}\cdot\vec{S}_2}{r^3} + \frac{G\left(\vec{S}_1\cdot\vec{S}_2 - 3\vec{r}\cdot\vec{S}_1\vec{r}\cdot\vec{S}_1/r^2\right)}{r^3}. \tag{13}$$

Note that since two spins are involved, there exist two spin–orbit coupling pieces, one for each particle, and a new spin–spin coupling piece arises.

The one-loop corrections to the non-relativistic scattering amplitude are rather lengthy and, due to the lack of space, we do not write them down in full. However, we find again that the spin-independent piece is identical to previous cases, while the spin–orbit terms possess the same form with the same numerical coefficients as in the spin-0–spin-1/2 and the spin-0–spin-1 calculations. The spin–spin piece is new

$$M^{SS} = G^2(m_1+m_2)\left(\frac{m_1m_2}{(m_1+m_2)^2}\frac{m_1m_2}{\vec{p}^2} + 9\right)\frac{\pi^2}{\sqrt{-q^2}}(q\cdot S_1 q\cdot S_2 - q^2 S_1\cdot S_2) + G^2\log -q^2\left(-\frac{11}{15}q\cdot S_1 q\cdot S_2 + \frac{16}{15}q^2 S_1\cdot S_2\right). \tag{14}$$

We observe that the spin–spin coupling piece of the amplitude contains a classical component proportional to $1/\vec{p}^2$ while the quantum spin–spin piece is well defined at threshold. The calculation of gravitational scattering of two spin-1/2 particles has also been performed recently by Kirilin [5] who only quotes the quantum component. For the spin–orbit quantum pieces we agree with Kirilin's results, but our spin–spin quantum terms differ by numerical coefficients. Our quantum correction to the spin–spin interaction potential is

$$V^{SS}_{NLO,\text{quantum}}(r) = \frac{G^2(43\vec{S}_1\cdot\vec{S}_2 - 55\vec{r}\cdot\vec{S}_1\vec{r}\cdot\vec{S}_1/r^2)}{10\pi r^5}. \tag{15}$$

4. Aside on QED scattering

In a parallel calculation we have also investigated the leading long distance effects in the electromagnetic scattering of two charged particles with different spin configurations, extending work by Feinberg and Sucher [8, 9]. Our results will be published in detail soon [7]. The similarity of the gravitational scattering results and the QED results is striking. We find universality in the QED case for spin-independent, spin–orbit, spin–spin and quadrupole pieces of the amplitudes/potentials to 1-loop. Furthermore, both classical and quantum corrections arise, and pieces proportional to $1/\vec{p}^2$ appear for all classical corrections except those which are spin independent, while the quantum pieces are all well behaved at threshold.

5. Conclusions

We have calculated the leading long distance corrections to the non-relativistic gravitational scattering amplitude for various spins of the scattered particles. The corrections involve classical as well as quantum contributions. Most importantly, we have explicitly shown that the corrections to the spin-independent and to the spin–orbit pieces are universal, i.e., the forms and coefficients are independent of the specific spin of the scattered particles. We suspect this universality to hold for arbitrary spin and also to hold for the spin–spin component.

Corrections to the Newtonian potential were obtained by Fourier transforming the scattering amplitude, but there is no unique definition of a scattering potential in quantum field theory. We will elaborate on these issues in more detailed publications [7]. The issue of the definition of a potential is also related to the classical spin-dependent corrections to the scattering amplitude which show an interesting threshold behaviour proportional to $1/\vec{p}^2$ and which, when subtracted, allow the construction of a second-order potential that can be used in a Schrödinger equation. Nevertheless, these pieces are of classical origin and they do occur in the scattering amplitude, leaving us to wonder if they can play any phenomenological role in gravitational wave experiments where spin-dependent post-Newtonian corrections are important.

Acknowledgments

AR would like to thank the organizers of IRGAC 2006 for an interesting and well-organized meeting, and we thank John Donoghue for valuable discussions. This work has been supported in part by the US National Science Foundation under award PHY-05-53304.

References

[1] Donoghue J F 1994 *Phys. Rev. Lett.* **72** 2996 (*Preprint* gr-qc/9310024)
[2] Donoghue J F 1994 *Phys. Rev.* D **50** 3874 (*Preprint* gr-qc/9405057)
[3] Donoghue J F 1995 *Preprint* gr-qc/9512024
[4] Bjerrum-Bohr N E J, J F Donoghue and Holstein B R 2003 *Phys. Rev.* D **67** 084033 (*Preprint* hep-th/0211072)
Bjerrum-Bohr N E J, J F Donoghue and Holstein B R 2005 *Phys. Rev.* D **71** 069903 (erratum)
[5] Kirilin G G 2005 *Nucl. Phys.* B **728** 179 (*Preprint* gr-qc/0507070)
[6] Sucher J 1994 *Preprint* hep-ph/9412388
[7] Ross A and Holstein B R in preparation
[8] Feinberg G and Sucher J 1988 *Phys. Rev.* D **38** 3763
Feinberg G and Sucher J 1991 *Phys. Rev.* D **44** 3997 (erratum)
[9] Feinberg G and Sucher J 1992 *Phys. Rev.* D **45** 2493